Table 1 Comparison of present solution with Bartz's solution

τ, sec	$X=0.0$	$X=0.2$	$X=0.4$	$X=0.6$	$X=0.8$	$X=1.0$	θ_m at outer surface	h_p,[a] W/m^2-K	h_B,[b] W/m^2-K	ϵ[c]
	θ_c									
6	0.29505	0.16941	0.08610	0.03859	0.01616	0.00981	0.00983	1821.9	2254.2	19.176
7	0.31093	0.18719	0.10164	0.04987	0.02373	0.01596	0.01588	1810.0	2254.2	19.705
8	0.29965	0.18677	0.10657	0.05607	0.02941	0.02122	0.02116	1610.3	2254.2	28.562
9	0.32442	0.20891	0.12459	0.06965	0.03960	0.03016	0.03020	1690.9	2254.2	24.991
10	0.33400	0.22077	0.13650	0.08022	0.04867	0.03860	0.03855	1669.7	2254.2	25.929
11	0.34157	0.23088	0.14725	0.09033	0.05782	0.04732	0.04724	1641.9	2254.2	27.162
12	0.33017	0.22716	0.14860	0.09446	0.06315	0.05296	0.05291	1497.6	2254.2	33.563
13	0.33116	0.23173	0.15521	0.10188	0.07073	0.06052	0.06046	1443.1	2254.2	35.983
14	0.33094	0.23509	0.16080	0.10858	0.07783	0.06771	0.06764	1387.0	2254.2	38.470
15	0.34418	0.24817	0.17321	0.12008	0.08857	0.07815	0.07823	1413.0	2254.2	37.316
16	0.34751	0.25380	0.18027	0.12785	0.09661	0.08625	0.08615	1383.7	2254.2	38.617

[a] h_p = heat-transfer coefficient (present). [b] h_B = heat-transfer coefficient (Bartz). [c] ϵ = percentage error $\{(h_B - h_p)/h_B\} \times 100$.

convergence is assumed. Upon further examination of Eqs. (5), it is found that, for small values of *Bo*, $\partial^2\theta/\partial Bo^2 \approx (\partial\theta/\partial Bo)^2$. Thus, if a small initial value of *Bo* is used, the convergence criterion for rapid convergence becomes

$$|Bo| < 1.5708 \tag{11}$$

The computer technique for solving for a value of *Bo* for a particular location and time is as follows. Start with a small initial value of *Bo*, satisfy the convergence criterion, and untilize the Newton-Raphson method to solve for the correct value of *Bo*.

Example

Using the present method, estimation of convective heat-transfer coefficient is carried out in conjunction with the experimental data of outer surface temperatures of M.S. used for nozzle divergent in a rocket motor static test. Insulated chromel/alumel thermocouples of 30 gage were used for measuring the temperature. The nozzle conditions and material properties taken are $b=0.0211$m, $T_o=300$ K, $\rho=7900$ kgm^3, $C_p=545$ W-sec/kg−K, $T_g=2946.2$ K, thermal conductivity (average) = 35 W/mK, and burning time = 16 sec. It is seen from Table 1 that estimated values of the convective heat-transfer coefficient are somewhat lower than the calculated results of Bartz.[12] Thus, it shows that Bartz's equation gives conservative estimation for the convective heat transfer. This is also is demonstrated experimentally by Brinsmade and Desmon.[13]

Conclusions

The Newton-Raphson iteration procedure proves quite useful in estimating the value of the heat-transfer coefficient from temperature data measured on the outer surface of the nozzle. The convergence criteria for the iteration are indicated. Extension of the method for two dimensions is straightforward. It has the advantage that temperatures can be found directly at specified time and location whereas the numerical approach requires the development of the temperature profile from the initial state.

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Three-Dimensional Moisture Diffusion in Laminated Composites

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Introduction

THE mathematics of moisture diffusion in laminated composites are based on Fick's analogy to Fourier's heat conduction equation.[1] The application of Fick's Law to one-dimensional, through-the-thickness moisture absorption in composite laminates has been established by Shen and

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Springer.[2] They also presented a method for determining an effective one-dimensional diffusion coefficient which would account for absorption along the edges of a test specimen.

In the present Note, a three-dimensional form of Fick's Law is applied to moisture absorption in laminated fiber reinforced composite materials, and a solution obtained for a special case of practical interest. Numerical results from the three-dimensional solution are compared to the approximate one-dimensional solution. Theoretical results on total weight gain as a function of time are compared to experimental data obtained on laminates fabricated from Hercule's AS/3501-5 graphite/epoxy system.

Three-Dimensional Solution

For a laminated plate in which the in-plane diffusion properties are orthotropic with respect to an x_1, x_2 axis system, with x_3 denoting the thickness coordinate. Fick's Law takes the form[1]

$$\frac{\partial m}{\partial t}=\bar{d}_{11}\frac{\partial^2 m}{\partial x_1^2}+\bar{d}_{22}\frac{\partial^2 m}{\partial x_2^2}+\bar{d}_{33}\frac{\partial^2 m}{\partial x_3^2} \tag{1}$$

where m is the moisture concentration expressed in terms of percent weight gain per unit volume, and $\bar{d}_{ii}$ are effective laminate diffusivities as defined by Shen and Springer.[2]

Consider a rectangular laminated plate subjected to the conditions

$$m=m_0\ ,\quad t\leq 0 \tag{2a}$$

$$m(\pm a_1/2,x_2,x_3,t)=m(x_1,\pm a_2/2,x_3,t)$$
$$=m(x_1,x_2,\pm a_3/2,t)=m_\infty\ ,\quad t>0 \tag{2b}$$

where m_0 is an initial uniform weight gain, m_∞ is the long time equilibrium weight gain, and a_1,a_2,a_3 correspond to the plate dimensions parallel to the x_1,x_2,x_3 directions, respectively.

A solution to Eq. (1) which satisfies the conditions of Eq. (2) takes the form[1]

$$m=m_0+(m_\infty-m_0)g \tag{3}$$

where

$$g=1-m_1(x_1,t)m_2(x_2,t)m_3(x_3,t) \tag{4}$$

and $m_i(x_i,t)$ are solutions to the one-dimensional equation

$$\frac{\partial m_i}{\partial t}=\bar{d}_{ii}\frac{\partial^2 m_i}{\partial x_i^2} \tag{5}$$

subjected to the conditions

$$m_i(x_i,t)=1\ ,\quad t\leq 0 \tag{6a}$$

$$m_i(\pm a_i/2,t)=0\ ,\quad t>0 \tag{6b}$$

Integration of Eq. (3) over the plate volume yields the total percent weight gain with the result

$$M=M_0+(M_\infty-M_0)G \tag{7}$$

where

$$G=1-M_1(t)M_2(t)M_3(t) \tag{8}$$

and

$$M_i(t)=\int_{-a_i/2}^{a_i/2} m_i(x_i,t)\,\mathrm{d}x_i \tag{9}$$

The functions m_i can be obtained by classical separation of variables or by the Laplace transform method.[1] Separation of variables yields a Fourier series solution, whereas the Laplace transform method yields a solution in terms of an infinite series of error function complements.

Short Time Solutions

In the case of total weight gain, it can easily be shown that for short times, only the first two terms of the error function complement solution need be retained with the following results for the three-dimensional case

$$G=1-\left(1-4\sqrt{\frac{t_1^*}{\pi}}\right)\left(1-\sqrt{\frac{t_2^*}{\pi}}\right)\left(1-4\sqrt{\frac{t_3^*}{\pi}}\right),\quad t_i^*<0.1 \tag{10}$$

where

$$t_i^*=\bar{d}_{ii}t/a_i^2 \tag{11}$$

Equation (10) is expanded form yields a third-order polynomial in $\sqrt{t}$. For $t_i^*<0.005$, only the first-order term need be retained with the result

$$G=\frac{4}{\sqrt{\pi}}\left(\frac{\sqrt{\bar{d}_{11}}}{a_1}+\frac{\sqrt{\bar{d}_{22}}}{a_2}+\frac{\sqrt{\bar{d}_{33}}}{a_3}\right)\sqrt{t} \tag{12}$$

Thus, for a short time period, G varies linearly with $\sqrt{t}$. Equation (12) can also be written in the form

$$G=4/a_3\sqrt{t/\pi}\sqrt{\bar{d}} \tag{13}$$

where

$$\bar{d}=\bar{d}_{33}\left(1+\frac{a_3}{a_1}\sqrt{\frac{\bar{d}_{11}}{\bar{d}_{33}}}+\frac{a_3}{a_2}\sqrt{\frac{\bar{d}_{22}}{\bar{d}_{33}}}\right)^2 \tag{14}$$

It should be noted that Eq. (14) is of the same form as used by Shen and Springer[2] for an edge correction factor to be utilized in conjunction with a one-dimensional solution for total weight gain. In particular, for moderately thick plates they suggested using $\bar{d}$ in conjunction with a one-dimensional solution to approximate a three-dimensional solution.

Results

Consider a rectangular laminate with in-plane dimensions a, b and thickness h constructed from plies of the same unidirectional material having symmetric fiber packing and laminate stacking geometries which yield $\bar{d}_{11}=\bar{d}_{22}$. A cursory examination of the transformation equations for diffusivity[2] shows the in-plane coefficients to be independent of any rotation in the x_1, x_2 plane. Thus, the laminate is quasi-isotropic with respect to the in-plane diffusivities. Using the micromechanics expressions along with the effective diffusivities presented by Shen and Springer[2] yields

$$\bar{d}_{11}=\bar{d}_{22}=\left(\frac{1-\sqrt{V_f\pi}-V_f/2}{1-\sqrt{V_f/\pi}}\right)\mathrm{d}_T \tag{15}$$

where V_f is the fiber volume fraction, which is chosen as 0.6 in this Note, and d_T is the diffusivity of a unidirectional composite transverse to the fibers. Because of the symmetry of fiber packing in each unidirectional ply, $\bar{d}_{33}=d_T$. Equation (15) yields the result

$$\bar{d}_{11}/d_T=\bar{d}_{22}/d_T=2.09 \tag{16}$$

Numerical results for total weight gain as a function $\sqrt{t^*}$ are shown in Fig. 1 where $t^*=d_Tt/h^2$. As indicated by Eqs. (13) and (14), the three-dimensional solution and the one-dimensional approximation yield identical results for short times. For longer times, the one-dimensional approximation overestimates the diffusion rate, as interaction between diffusion through the edges and diffusion through the thickness slows down the apparent three-dimensional diffusion process. As anticipated, the one-dimensional solution

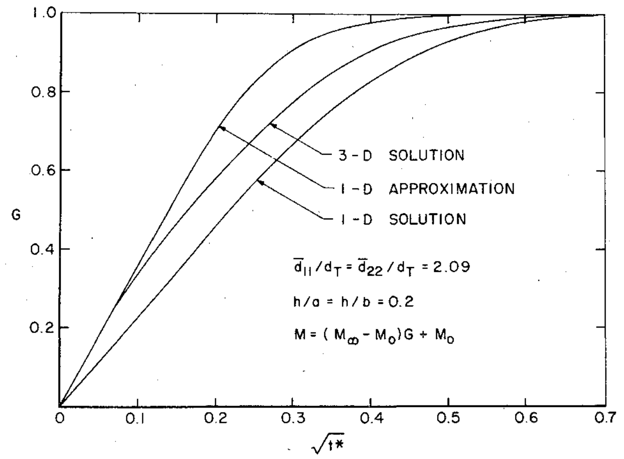

Fig. 1 Total weight gain for three-dimensional diffusion in a thick composite laminate.

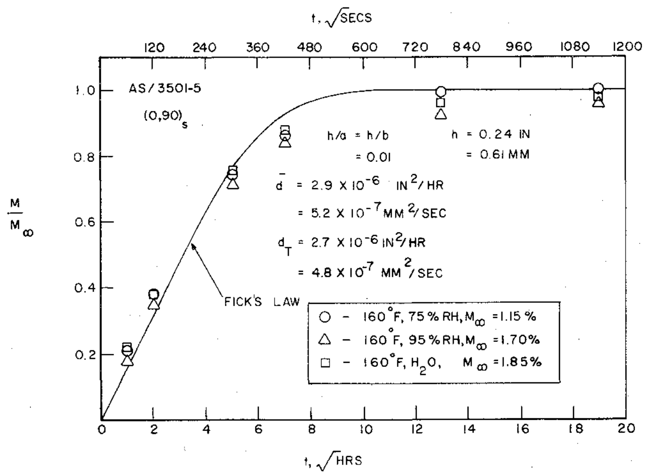

Fig. 2 Comparison of theory and experiment for a thin graphite/epoxy composite laminate.

underestimates the diffusion process because of the edge effects.

In order to assess the accuracy of the three-dimensional form of Fick's Law, two laminated plates were fabricated from Hercule's AS/3501-5 graphite/epoxy pre-preg system. One panel was a four-ply $(0,90)_s$ construction from which 2×2 in. specimens were cut. The second panel was constructed of 20 plies with the stacking geometry $(0,90)_{5s}$ from which 1-in.×1-in. specimens were cut. Thus, the first panel provided thin specimens for measuring $\bar{d}_{33}$, while the second panel provided thick specimens for measuring three-dimensional diffusion. A number of specimens were placed on racks in environmental chambers with the temperature maintained at 160°F and the relative humidity controlled at either 75% or 95%. The remaining specimens were placed in distilled water in the environmental chamber having 85% RH and a 160°F temperature. All specimens were dried out in a vacuum oven at 200°F and weighted prior to environmental conditioning. Specimens were withdrawn at various times from the environmental chamber and their wet weight measured and recorded.

A fit of the experimental data to Fick's Law for the thin specimens is displayed in Fig. 2 where percent weight gain is plotted against $\sqrt{t}$. A small edge correction factor of about 6% was used to determine d_T. Thus, for all practical purposes, the diffusion process for these laminates was one-dimensional in nature. The value of d_T determined in Fig. 2 was used in conjunction with Eq. (16) and various solutions of Fick's Law to predict the precent weight gain as a function of $\sqrt{t}$ for the

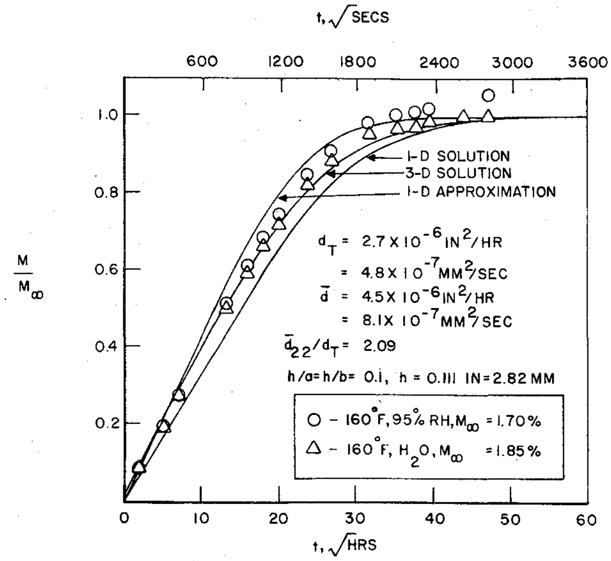

Fig. 3 Comparison of theory and experiment for a moderately thick graphite/epoxy laminate.

thick specimens. Comparisons between theory and experiment are shown in Fig. 3. Each data point in both Figs. 2 and 3 is an average of 3 to 5 specimens.

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Similar Solutions for an Axisymmetric Laminar Boundary Layer on a Circular Cylinder

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SIMILAR solutions of the equations describing steady, constant-density, constant-viscosity flow in laminar boundary layers are well documented for plane two-dimensional flow.[1-5] A further example of some interest is that for axisymmetric flow on a circular cylinder. Along a continuous, stationary cylindrical surface, x is measured in the direction of the generators, and, perpendicular to this, a curvilinear coordinate z is drawn on the surface. A coordinate y is measured into the flow along straight lines normal to the surface. The curvature of the surface in the y-z plane is $K(z)$ (convex positive). Velocity components in the x and y directions are, respectively, u and v; there is no velocity in the z direction.

Of the complete Navier-Stokes equations,[6] that for the x direction is differentiated with respect to y and then subtracted from the x derivative of the equation for the y direction. This yields a single equation of motion from which

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